

Global regularity for a model Navier-Stokes equations on \mathbb{R}^3

Dongho Chae

Department of Mathematics
Chung-Ang University
Seoul 156-756, KOREA
e-mail: *dchae@cau.ac.kr*

Abstract

We study a nonlinear parabolic system for a time dependent solenoidal vector field on \mathbb{R}^3 . The nonlinear term of this new model equations is obtained slightly modifying that of the Navier-Stokes equations. The system has the same scaling property and the Galileian invariance as the Navier-Stokes equations. For such system we prove the global regularity for a smooth initial data.

AMS Subject Classification Number: 35K55, 35B05, 76A02

1 Introduction

We consider the incompressible Navier-Stokes equations in \mathbb{R}^3 ,

$$(NS) \begin{cases} v_t + v \cdot \nabla v = -\nabla p + \Delta v, & (x, t) \in \mathbb{R}^3 \times [0, \infty) \\ \nabla \cdot v = 0, & (x, t) \in \mathbb{R}^3 \times [0, \infty) \\ v(x, 0) = v_0(x), & x \in \mathbb{R}^3 \end{cases}$$

where $v = (v_1, v_2, v_3)$, $v_j = v_j(x, t)$, $j = 1, 2, 3$, is the velocity field, and $p = p(x, t)$ is the pressure. For simplicity we consider the case of zero external force. The problem of global regularity/finite time singularity of solutions to

(NS) for a smooth initial data v_0 is an outstanding open problem. We know the local in time well-posedness for smooth initial data([14, 10]), the global existence of weak solutions([13, 9]), and the partial regularity of suitable weak solutions([18, 1]). For comprehensive studies of the Cauchy problem of (NS) we refer [11, 12]. Using the vector identity

$$v \cdot \nabla v = -v \times \operatorname{curl} v + \frac{1}{2} \nabla |v|^2,$$

One can rewrite the system (NS) in an equivalent form,

$$(NS)_1 \begin{cases} v_t - v \times \omega = -\nabla(p + \frac{1}{2}|v|^2) + \Delta v, \\ \nabla \cdot v = 0, \omega = \nabla \times v \\ v(x, 0) = v_0(x). \end{cases}$$

For further discussion let us recall the definition of the Riesz transform([19]) in \mathbb{R}^n , $n \in \mathbb{N}$. For $f \in L^p(\mathbb{R}^n)$, $p \in [1, \infty)$, the Riesz transform R_j of f is given by

$$R_j(f)(x) = c_n \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} \frac{y_j}{|y|^{n+1}} f(x-y) dy, \quad j = 1, \dots, n,$$

where

$$c_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}}.$$

Let $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) dx$ be the Fourier transform of f . Then, the Riesz transform is more conveniently defined by the Fourier transform as

$$(\widehat{R_j f})(\xi) = i \frac{\xi_j}{|\xi|} \hat{f}(\xi), \quad i = \sqrt{-1}. \quad (1.1)$$

Let R_1, R_2, R_3 be the Riesz transforms in \mathbb{R}^3 , and $u = (u_1, u_2, u_3)$ be a vector field on \mathbb{R}^3 . Then, we define for scalar function f , $R(f) := (R_1(f), R_2(f), R_3(f))$, and for a vector field $u = (u_1, u_2, u_3)$,

$$R \cdot u := R_1(u_1) + R_2(u_2) + R_3(u_3),$$

and

$$R \times u := (R_2 u_3 - R_3 u_2, R_3 u_1 - R_1 u_3, R_1 u_2 - R_2 u_1).$$

Then, using the fact $R_1^2 + R_2^2 + R_3^2 = -I$, which follows immediately from (1.1), we have for a vector field F , satisfying $\operatorname{div} F = 0$,

$$R \times R \times F = R^2 F - R(R \cdot F) = -F.$$

Since

$$R \times R \times \left\{ \nabla \left(p + \frac{1}{2} |v|^2 \right) \right\} = 0, \quad R \times R \times \{v_t - \Delta v\} = -v_t + \Delta v$$

for v with $\operatorname{div} v = 0$, the system $(NS)_1$ can be written as the following equivalent system.

$$(NS)_2 \begin{cases} v_t + R \times R \times (v \times \omega) = \Delta v, \\ v(x, 0) = v_0(x), \quad \nabla \cdot v_0 = 0. \end{cases}$$

The implication that $(NS)_2$ follows from $(NS)_1$ is obvious by taking $R \times R \times$ on $(NS)_1$. For the reverse direction we note that the first equation of $(NS)_2$ can be written as

$$R \times R \times (v_t - v \times \omega - \Delta v) = 0,$$

from which it follows that there exists a scalar function $p = p(x, t)$ such that

$$v_t - v \times \omega - \Delta v = -\nabla \left(p + \frac{1}{2} |v|^2 \right).$$

Note that the divergence free condition of the initial data is preserved by the equations in $(NS)_2$ for smooth solutions, and the solution satisfies $\operatorname{div} v = 0$ automatically. We also observe that in the formulation $(NS)_2$ there exists no pressure term, although the nonlocality is now moved to the nonlinear term via the Riesz transforms.

We expect that the problem of the global regularity/finite time singularity for the system $(NS)_2$ has the similar difficulty as the original Navier-Stokes equations (NS) . Instead of $(NS)_2$ we consider its modified version:

$$(mNS) \begin{cases} v_t + R \times (v \times \omega) = \Delta v, \\ v(x, 0) = v_0(x), \quad \nabla \cdot v_0 = 0, \end{cases}$$

which is obtained by omitting one nonlocal operation $R \times$ in the nonlinear term in the system $(NS)_2$. Another interpretation of (mNS) is its relation with the following Hall equations, which is obtained from the Hall

magnetohydrodynamics(Hall-MHD) equations by setting the velocity= 0, and which represent the major difficult part of the whole system(see [2, 3, 4, 5] for more detailed studies of the Hall-MHD system).

$$(Hall) \begin{cases} B_t + \nabla \times (B \times (\nabla \times B)) = \Delta B, & (x, t) \in \mathbb{R}^3 \times (0, \infty) \\ B(x, 0) = B_0(x), \quad \nabla \cdot B_0 = 0, & x \in \mathbb{R}^3 \end{cases}$$

where $B = (B_1, B_2, B_3)$, $B_j = B_j(x, t)$, $j = 1, 2, 3$ is the magnetic field. Comparing with (Hall), we find that the nonlinear term of (mNS) is regularized by one derivative, in the sense that $\nabla \times$ in front of the nonlinear term (Hall) is replaced by $R \times = \nabla(-\Delta)^{-1/2} \times$ in (mMS). We note that the symmetry properties(say, the Galilean invariance and the scaling symmetry) of (mNS) are the same as the original Navier-Stokes equations. Our result in the following theorem shows that for such modified system (mNS) we could show the global regularity for a given smooth initial data.

Theorem 1.1 *Let $v_0 \in H^m(\mathbb{R}^3)$ with $m > 5/2$. Then, for all $T \in (0, \infty)$ there exists a unique solution $v \in C([0, T]; H^m(\mathbb{R}^3)) \cap L^2(0, T; H^{m+1}(\mathbb{R}^3))$ to the system (mNS) . Moreover, the solution satisfies the inequality:*

$$\begin{aligned} & \sup_{0 < t < T} \|v(t)\|_{H^m}^2 + \int_0^T \|Dv(s)\|_{H^m}^2 ds \\ & \leq \|v_0\|_{H^m}^2 \exp \left\{ T \|v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \right\} \times \\ & \times \exp \left[\|\Lambda v_0\|_{L^2}^2 \exp \left\{ CT \|v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \|\Lambda v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \right\} \right] \end{aligned} \quad (1.2)$$

for all $T > 0$.

Remark 1.1 The result shows that the simple estimates for the nonlinear term of $(NS)_2$ is not enough to deduce correct result for the problem of the global regularity/finite time singularity. We mention here that there are also studies of the other model equations of the Navier-Stokes system, where the authors show the finite time blow-up (see e.g. [6, 7, 8, 16, 17, 20]).

2 Proof of Theorem 1.1

Let us set the multi-index $\alpha := (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$ with $|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$. Then, $D^\alpha := D_1^{\alpha_1} D_2^{\alpha_2} \dots D_n^{\alpha_n}$, where $D_j = \partial/\partial x_j$,

$j = 1, 2, \dots, n$. Given $m \in \mathbb{N} \cup \{0\}$ the Sobolev space, $H^m(\mathbb{R}^n)$ is the Hilbert space of functions consisting of functions $f \in L^2(\mathbb{R}^n)$ such that

$$\|f\|_{H^m} := \left(\sum_{|\alpha| \leq m} \int_{\mathbb{R}^n} |D^\alpha f(x)|^2 dx \right)^{\frac{1}{2}} < \infty,$$

where the derivatives are in the sense of distributions. Given $s \in \mathbb{R}$, we use the notation $\Lambda^s(f) = (-\Delta)^{\frac{s}{2}} f$, which is defined by its Fourier transform as

$$\widehat{\Lambda^s(f)}(\xi) = |\xi|^s \hat{f}(\xi).$$

Then, we observe that

$$\widehat{R_j(f)} = i \frac{\xi_j}{|\xi|} \hat{f} = \partial_j \widehat{\Lambda^{-1}f}(\xi).$$

We use the energy method for the proof of Theorem 1.1 (see e.g. [15]).

Proof of Theorem 1.1 Since the local well-posedness in the Sobolev space $H^m(\mathbb{R}^3)$ for $m > 5/2$ is standard, we proceed directly to the global in time a priori estimate. We take L^2 inner product (mNS) by Λv to obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Lambda^{\frac{1}{2}} v\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 &= - \int_{\mathbb{R}^3} R \times (v \times \omega) \cdot \Lambda v \, dx \\ &= - \int_{\mathbb{R}^3} (v \times \omega) \cdot \Lambda R \times v \, dx = - \int_{\mathbb{R}^3} (v \times \omega) \cdot \omega \, dx = 0. \end{aligned}$$

We have therefore

$$\frac{1}{2} \|\Lambda^{\frac{1}{2}} v(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\frac{3}{2}} v(s)\|_{L^2}^2 ds = \frac{1}{2} \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2 \quad (2.1)$$

Next we take L^2 inner product (mNS) by v to deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2 &= - \int_{\mathbb{R}^3} v \times \omega \cdot R \times v \, dx \\ &\leq \int_{\mathbb{R}^3} |v| |\omega| |Rv| \, dx \leq \|v\|_{L^6} \|\omega\|_{L^3} \|Rv\|_{L^2} \\ &\leq C \|\nabla v\|_{L^2} \|\Lambda^{\frac{3}{2}} v\| \|v\|_{L^2} \\ &\leq \frac{1}{2} \|\nabla v\|_{L^2}^2 + C \|\Lambda^{\frac{3}{2}} v\|^2 \|v\|_{L^2}^2. \end{aligned}$$

Hence, we have

$$\begin{aligned}\|v(t)\|_{L^2}^2 + \int_0^t \|\nabla v\|_{L^2}^2 ds &\leq \|v_0\|_{L^2}^2 \exp\left(C \int_0^t \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 ds\right) \\ &\leq \|v_0\|_{L^2}^2 \exp(C\|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2),\end{aligned}\quad (2.2)$$

where we used the estimate (2.1). We now take L^2 inner product (mNS) by Δv to have

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|\Lambda v\|_{L^2}^2 + \|\Delta v\|_{L^2}^2 &= - \int_{\mathbb{R}^3} v \times \omega \cdot R \times \Delta v \, dx \\ &\leq \|v\|_{L^6} \|\omega\|_{L^3} \|R \Delta v\|_{L^2} \leq C \|\nabla v\|_{L^2} \|\Lambda^{\frac{3}{2}} v\|_{L^2} \|\Delta v\|_{L^2} \\ &\leq \frac{1}{2} \|\Delta v\|_{L^2}^2 + C \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 \|\nabla v\|_{L^2}^2,\end{aligned}$$

from which we obtain

$$\begin{aligned}\|\Lambda v(t)\|_{L^2}^2 + \int_0^t \|\Delta v\|_{L^2}^2 ds &\leq \|\Lambda v_0\|_{L^2}^2 \exp\left(C \int_0^t \|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 ds\right) \\ &\leq \|\Lambda v_0\|_{L^2}^2 \exp(C\|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2)\end{aligned}\quad (2.3)$$

by (2.1). We operate D^2 on (mNS), and take L^2 inner product of it by $D^2 v$ to obtain

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \|D^2 v\|_{L^2}^2 + \|D^3 v\|_{L^2}^2 &= - \int_{\mathbb{R}^3} D(R \times (v \times \omega)) \cdot D^3 v \, dx \\ &= - \int_{\mathbb{R}^3} (Dv \times \omega + v \times D\omega) \cdot R \times D^3 v \, dx \\ &\leq (\|Dv\|_{L^3} \|\omega\|_{L^6} + \|v\|_{L^\infty} \|D\omega\|_{L^2}) \|D^3 v\|_{L^2} \\ &\leq C(\|\Lambda^{\frac{3}{2}} v\|_{L^2} + \|v\|_{L^\infty}) \|D^2 v\|_{L^2} \|D^3 v\|_{L^2} \\ &\leq \frac{1}{2} \|D^3 v\|_{L^2}^2 + C(\|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 + \|v\|_{L^\infty}^2) \|D^2 v\|_{L^2}^2.\end{aligned}$$

Hence,

$$\begin{aligned}\|D^2 v(t)\|_{L^2}^2 + \int_0^t \|D^3 v\|_{L^2}^2 ds &\leq \|D^2 v_0\|_{L^2}^2 \exp\left\{C \int_0^t (\|\Lambda^{\frac{3}{2}} v\|_{L^2}^2 + \|v\|_{L^\infty}^2) ds\right\} \\ &\leq \|\Lambda v_0\|_{L^2}^2 \exp\left\{C \int_0^t (\|v(s)\|_{L^2}^2 + \|\Delta v(s)\|_{L^2}^2) ds\right\} \\ &\leq \|\Lambda v_0\|_{L^2}^2 \exp\left\{Ct \|v_0\|_{L^2}^2 \exp(C\|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \|\Lambda v_0\|_{L^2}^2 \exp(C\|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2)\right\},\end{aligned}\quad (2.4)$$

where we used the estimates (2.2) and (2.3). Let $m > 5/2$. Operating D^α on (mNS) and taking L^2 inner product of it by $D^\alpha v$, and summing over $|\alpha| \leq m$, one has

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|v\|_{H^m}^2 + \|Dv\|_{H^m}^2 &= - \sum_{|\alpha| \leq m} \int_{\mathbb{R}^3} D^\alpha (v \times \omega) \cdot R \times D^\alpha v \, dx \\
&= \sum_{|\beta|=|\alpha|-1, 1 \leq |\alpha| \leq m} \int_{\mathbb{R}^3} D^\beta (v \times \omega) \cdot R \times D^\alpha Dv \, dx - \int_{\mathbb{R}^3} (v \times \omega) \cdot R \times v \, dx \\
&\leq C \|v \times \omega\|_{H^{m-1}} \|RDv\|_{H^m} \leq C (\|v\|_{L^\infty} + \|\nabla v\|_{L^\infty}) \|v\|_{H^m} \|Dv\|_{H^m} \\
&\leq \frac{1}{2} \|Dv\|_{H^m}^2 + C (\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty}^2) \|v\|_{H^m}^2,
\end{aligned}$$

and therefore

$$\begin{aligned}
\|v(t)\|_{H^m}^2 + \int_0^t \|Dv(s)\|_{H^m}^2 \, ds &\leq \|v_0\|_{H^m}^2 \exp \left(C \int_0^t (\|v\|_{L^\infty}^2 + \|\nabla v\|_{L^\infty}^2) \, ds \right) \\
&\leq \|v_0\|_{H^m}^2 \exp \left\{ C \int_0^t (\|v\|_{L^2}^2 + \|D^3 v\|_{L^2}^2) \, ds \right\} \\
&\leq \|v_0\|_{H^m}^2 \exp \left\{ t \|v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \right\} \times \\
&\quad \times \exp \left[\|\Lambda v_0\|_{L^2}^2 \exp \left\{ Ct \|v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \|\Lambda v_0\|_{L^2}^2 \exp(C \|\Lambda^{\frac{1}{2}} v_0\|_{L^2}^2) \right\} \right],
\end{aligned} \tag{2.5}$$

where we used the estimates (2.2) and (2.4). \square

Acknowledgements

This work was supported partially by NRF Grant no. 2006-0093854 and 2009-0083521.

References

- [1] L. Caffarelli, R. Kohn and L. Nirenberg, *Partial Regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math., **35**, (1982), pp. 771-831.
- [2] D. Chae, P. Degond and J-G. Liu, *Well-posedness for Hall-magnetohydrodynamics*, Ann. Inst. Henri Poincaré-Analyse Nonlineaire, **31**, (2014), pp. 555-565.

- [3] D. Chae and J. Lee, *On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics*. J. Differential Equations, **256**, (2014), no. 11, pp. 3835-3858.
- [4] D. Chae and M. Schonbek. *On the temporal decay for the Hall-magnetohydrodynamic equations*. J. Differential Equations, **255**, (2013), no. 11, pp. 3971-3982.
- [5] D. Chae and S. Weng, *Singularity formation for the incompressible Hall-MHD equations without resistivity*, Ann. Inst. Henri Poincaré-Analyse Nonlineaire, in print.
- [6] A. Cheskidov, *Blow-up in finite time for the dyadic model of the Navier-Stokes equations*, Trans. Am. Math. Soc. , **360**, (2010), pp. 5101-5120.
- [7] S. Friedlander and N. Pavlovic, *Remarks concerning modified Navier-Stokes equations*, , Discrete Contin. Dyn. Syst., **10**, (2004), no. 1-2, pp. 269-288.
- [8] I. Gallagher and M. Paicu, *Remarks on the blow-up of solutions to a toy model for the Navier-Stokes equations*, Proc. Amer. Math. Soc., **137**, no. 6, (2009), pp. 2075-2083.
- [9] E. Hopf, *Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen*, Math. Nacht., **4**, (1951), pp. 213-231.
- [10] T. Kato, *Nonstationary flows of viscous and ideal fluids in \mathbb{R}^3* , J. Funct. Anal., **9**, (1972), pp. 296-305.
- [11] O. A. Ladyzenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, (1969).
- [12] P. G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, research notes in mathematics series, **431**, Chapman & Hall/CRC , (2002).
- [13] J. Leray, *Essai sur le mouvement d'un fluide visqueux emplissant l'espace*, Acta Math., **63**, (1934), pp. 193-248.
- [14] L. Lichtenstein, *Über einige Existenzprobleme der Hydrodynamik homogener unzusammendrückbarer, reibungloser Flüssigkeiten und die*

- Helmholtzschen Wirbelsätze*, Math. Zeit., **23**, (1925), pp. 89-154; **26**, (1927), pp. 193-323, pp. 387-415 ; **32**, (1930), pp. 608-725.
- [15] A. J. Majda and A. L. Bertozzi, *Vorticity and Incompressible Flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press., (2002).
 - [16] S. Montgomery-Smith, *Finite time blow up for a Navier-Stokes like equation*, Proc. Amer. Math. Soc., **129** , no. 10, (2001), pp. 3025-3029.
 - [17] P. Plecháč and V. Sverák, *Singular and regular solutions of a nonlinear parabolic system*, Nonlinearity, **16**, no. 6, (2003), pp. 2083-2097.
 - [18] V. Schaffer, *Partial regularity of solutions to the Navier-Stokes equations*, Pacific J. Math. **66**, (1976), pp. 535-552.
 - [19] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J. 1970.
 - [20] T. Tao, *Finite time blowup for an averaged three-dimensional Navier-Stokes equations*, arXiv preprint no. 1402.0290.